

# A remark on completions of the class $\mathbf{S\mathfrak{Nr}_n CA_{n+k}}, k \geq 3$

Tarek Sayed Ahmed

Department of Mathematics, Faculty of Science,  
Cairo University, Giza, Egypt.

April 15, 2013

**Abstract .** We give a sufficient condition that implies that for  $k \geq n + 3$ , the class  $\mathbf{S\mathfrak{Nr}_n CA_{n+3}}$  is not closed under completions and is not Sahlqvist axiomatizable. We compare this condition to existing results in the literature.

We use a construction of Hirsch and Hodkinson [2]. We follow the notation in op.cit.  $\mathbf{RA}$  stands for the class of relation algebras and  $\mathbf{CA}_n$  stands for the class of cylindric algebras of dimension  $n$ .  $\mathbf{RaCA}_n$  stands for the class of relation algebra reducts of  $\mathbf{CA}_n$ .  $\mathbf{RCA}_n$  stands for the class of representable  $\mathbf{CA}_n$ 's and for  $k < n$ ,  $\mathbf{\mathfrak{Nr}_k CA_n}$  stands for the class of neat  $k$  reducts of algebras in  $\mathbf{CA}_n$ . It is known that  $\mathbf{RCA}_n = \mathbf{S\mathfrak{Nr}_n CA_\omega}$  for any finite  $n$  and that for  $k \in \omega$  and  $n > 2$ ,  $\mathbf{RCA}_n \subset \mathbf{S\mathfrak{Nr}_n CA_{n+k+1}} \subset \mathbf{S\mathfrak{Nr}_n CA_{n+k}}$  [2]. Let  $\mathbf{RA}_n$  be the class of subalgebras of atomic relation algebras having  $n$  dimensional relational basis. Then  $\mathbf{SRaCA}_n \subseteq \mathbf{RA}_n$  [2]. The full complex algebra of an atom structure  $S$  will be denoted by  $\mathfrak{Cm}S$ , and the term algebra by  $\mathfrak{Tm}S$ .  $S$  could be a relation atom structure or a cylindric atom structure.

**Theorem 1 .** *Let  $n \geq 3$ . Assume that for any simple atomic relation algebra  $A$  with atom structure  $S$ , there is a cylindric atom structure  $H$  such that:*

- (1) *If  $\mathfrak{Tm}S \in \mathbf{RRA}$ , then  $\mathfrak{Tm}H \in \mathbf{RCA}_n$ .*
- (2)  *$\mathfrak{Cm}S$  is embeddable in  $\mathbf{Ra}$  reduct of  $\mathfrak{Cm}H$ .*

*Then for all  $k \geq 3$ ,  $\mathbf{S\mathfrak{Nr}_n CA_{n+k}}$  is not closed under completions.*

**Proof.** Let  $S$  be a relation atom structure such that  $\mathfrak{Tm}S$  is representable while  $\mathfrak{Cm}S \notin \mathbf{RA}_6$ . Such an atom structure exists [2] Lemmas 17.34-17.36 and are finite. It follows that  $\mathfrak{Cm}S \notin \mathbf{SRaCA}_n$ . Let  $H$  be the  $\mathbf{CA}_n$  atom structure provided by the hypothesis of the previous theorem. Then  $\mathfrak{Tm}H \in \mathbf{RCA}_n$ . We claim that  $\mathfrak{Cm}H \notin \mathbf{S\mathfrak{Nr}_n CA_{n+k}}$ ,  $k \geq 3$ . For assume not, i.e. assume that

$\mathfrak{Cm}H \in \mathbf{S}\mathfrak{Nr}_n\mathbf{CA}_{n+k}$ ,  $k \geq 3$ . We have  $\mathfrak{Cm}S$  is embeddable in  $\mathbf{Ra}\mathfrak{Cm}H$ . But then the latter is in  $\mathbf{SRaCA}_6$  and so is  $\mathfrak{Cm}S$ , which is not the case. ■

**Corollary 2 .** *Assume the hypothesis in Theorem 1. Then the following hold:*

- (1) *There exist two atomic cylindric algebras of dimension  $n$  with the same atom structure, only one of which is representable.*
- (2) *For  $n \geq 3$  and  $k \geq 3$ ,  $\mathbf{S}\mathfrak{Nr}_n\mathbf{CA}_{n+k}$  is not closed under completions and is not atom-canonical. In particular,  $\mathbf{RCA}_n$  is not atom-canonical.*
- (3) *There exists a non-representable  $\mathbf{CA}_n$  with a dense representable subalgebra.*
- (4) *For  $n \geq 3$  and  $k \geq 3$ ,  $\mathbf{S}\mathfrak{Nr}_n\mathbf{CA}_{n+k}$  is not Sahlqvist axiomatizable. In particular,  $\mathbf{RCA}_n$  is not Sahlqvist axiomatizable.*
- (5) *There exists an atomic representable  $\mathbf{CA}_n$  with no complete representation.*

**Proof.** [3] ■

Monk and Maddux constructs such an  $H$  for  $n = 3$  and Hodkinson constructs an  $H$ , but  $H$  does not satisfy 2 [1].

## References

- [1] Hokinson *Constructing cylindric and polyadic algebras from atomic relation algebras*
- [2] Hirsch R., Hodkinson.I., *Relation algebras by games*. Studies in Logic and the Foundations of Mathematics. Volume 147.
- [3] Sayed Ahmed *The class ..is not closed under completions*